RELATIVELY UNIFORM CONVERGENCE AND THE CLASSIFICATION OF FUNCTIONS*

BY

E. W. CHITTENDEN

INTRODUCTION

The classification of functions of real variables given by Baire† is based upon iteration of simple limits. The classes of functions obtained are found to be closed relative to uniform convergence. Several types of convergence which are more exacting than simple convergence and less exacting (in general) than uniform convergence have proved useful in analysis, notably the sub-uniform convergence of Arzelà,‡ the uniform convergence in general of Weyl,§ and the relatively uniform convergence of E. H. Moore.|| It is of interest to inquire into the relation of these types of convergence to the Baire classification of functions. The present paper is concerned with the problem presented by the relatively uniform convergence of Moore which will be spoken of as convergence (R).

* Presented to the Society, September 8, 1920.

† Annali di Matematica, 1900.

‡ Arzelà, *Sulle serie di funzioni*, Memorie della Reale Accademia degli Scienze di Bologna, ser. 5, vol. 8 (1900). This type of convergence is called "convergenza uniforme a tratti" by Arzelà. The designation "sub-uniform" appears to have originated with E. H. Moore, Bulletin of the American Mathematical Society, vol. 7, p. 257.

§ H. Weyl, Mathematische Annalen, vol. 67 (1909), p. 225. Weyl has introduced under the designation, uniform convergence in general, the following type of convergence. Let a sequence of functions converge on a measurable set P. Then there is a set Q whose measure is as near the measure of P as you please on which the convergence is uniform.

If $e_1, e_2, \ldots, e_k, \ldots$ is a monotonic sequence of positive numbers with the limit zero, there will be sets

$$Q_1, Q_2, \ldots, Q_k, \ldots$$

such that mes $(P - Q_k) \leq e_k$, Q_{k+1} contains Q_k, and the convergence is uniform on each set Q_k. If $Q = \lim Q_k$ then $P - Q$ is of measure zero. It follows from the theorem of § 14 below that uniform convergence in general is equivalent to relatively uniform convergence excepting a set of points of measure null.

|| *Introduction to a form of general analysis, The New Haven Mathematical Colloquium,* Yale University Press, New Haven, 1910; p. 30, § 7d.

We are indebted to Baire for a detailed investigation of properties of functions of the classes 1, 2, 3,* but the general theory of functions of class α, where α denotes a transfinite number with at most a denumerable infinity of predecessors, was developed principally by Lebesgue, who was able, incidentally, to demonstrate the existence of functions of every class.†

Ch. J. de la Vallée-Poussin has given a very clear and concise account of the results obtained by Baire and Lebesgue in his monograph entitled *Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire.*‡ He has introduced some characteristic improvements in the exposition involving the discovery of new and powerful theorems, and exhibited the important rôle of the sets which are at once O and F of class α (that is ambiguous (A) of class α). The reader is referred to this monograph for the proofs of the theorems relating to the classification of Baire which are assumed in the following discussion. The notation and terminology of Vallée-Poussin are employed in this paper.

In previous papers the writer has shown that there exist sequences of functions which converge but do not converge uniformly relative to any scale function,§ and has found necessary and sufficient conditions that a function be the limit of a sequence of *continuous* functions convergent (R),|| thus completely characterizing the functions of what may be called the class R_1.

The present paper presents the results of further investigations on this subject. Let f denote a function of class α of Baire. If for every value of the real constant a the sets $(f > a)$, $(f < a)$ are ambiguous (A) of class α, the function f is said to be of class A_α, a proper subclass of class α. Denoting by R_α the class of functions defined in terms of convergence (R) as class α of Baire is defined in terms of simple convergence, then all functions of the class R_α which are not of class $<\alpha$ in the classification of Baire are contained in the class A_α. There is a subclass D_α of R_α (see §5) such that every function of class α is the limit of a *uniformly* convergent sequence of functions of D_α. If $\alpha = 1$, $R_\alpha = A_\alpha$. There are indications that the conditions which functions of the class R_α $(\alpha > 1)$ must satisfy are stronger than those imposed on the functions of A_α, but no examples demonstrating this are available.

A number of theorems dealing with the general theory of functions of class α have been found. These theorems are presented as they naturally occur in the development of the subject.

While the discussion is confined to functions defined on limited perfect sets in space of n dimensions this restriction is not essential, and it is possible to apply

* Acta Mathematica, vol. 30 (1906).

† Journal de Mathématiques, ser. 6, vol. 1 (1905).

‡ Gauthier-Villars, Paris, 1916.

§ These Transactions, vol. 15 (1914), pp. 197–201.

|| These Transactions, vol. 20 (1919), pp. 179–184.

the theory with suitable and fairly obvious modifications to classes of functions on a great variety of ranges. As an illustration we cite an article by the writer, *On a generalization of a theorem of Baire.**

I. Classification of functions and sets of points

1. **The Baire classification of functions.** The functions which enter the following discussion are assumed to be defined at every point of a fundamental limited perfect set P in space of n dimensions. The functional values are limited to the class of real numbers; that is, the functions are *finite*, completely defined, single-valued, real-valued functions of the variable point M with the range P.

Baire's classification of functions is defined as follows: the continuous functions are of class zero (B_0); the discontinuous functions which are limits of sequences of continuous functions are of class 1 (B_1), etc.; and in general the functions of class α (B_α) are those limit functions of denumerable sequences of functions of classes preceding class α which are not functions of class $< \alpha$.

2. **Lebesgue's classification of sets of points.** Lebesgue has introduced a classification of sets of points based on Baire's classification of functions which has proved a remarkably effective tool in the investigation of the properties of the classes B_α. A set E is said to be *open* or O of class α if there is a function Θ of class $\leqq \alpha$ such that E is the set of all points of P at which $\Theta > 0$. This set E is represented by the notation $E = (\Theta > 0)$. A set E is *closed* or F of class α if there is a function Θ of class $\leqq \alpha$ for which E is the set $E = (\Theta = 0)$.

Vallée-Poussin has called attention to the importance of the sets which are at once O and F of class α and has called them *ambiguous* or A of class α.

The proofs of the following statements regarding sets O, F, and A of class α will be found in the monograph of Vallée-Poussin already cited, §§ 132–134, pp. 132–139.

Every set O of class α is F of class $\alpha + 1$. Every set F of class α is O of class $\alpha + 1$. Therefore, every set which is either O or F of class α is A of class $\alpha + 1$.

We denote by Σ the operation of taking the sum of a finite or denumerably infinite sequence of sets, by Π the corresponding product. Then if O_α, F_α, A_α, signify sets which are O of class α, etc., we have

$$\sum O_\alpha = O_{\alpha+1}, \quad \prod F_\alpha = F_{\alpha+1},$$

and

$$\sum A_\alpha = A_{\alpha+1}, \quad \prod A_\alpha = A_{\alpha+1}.$$

---

* Bulletin of the American Mathematical Society, vol. 27 (1920–21), p. 5.

3. The limit of a sequence of sets of points. Given a sequence $E_1, E_2, E_3, \ldots, E_n, \ldots$ of sets of points we define the greatest limit of the sequence by the formula:

$$\overline{\lim}\, E_n = (E_1 + E_2 + E_3 + \cdots)(E_2 + E_3 + \cdots)(E_3 + \cdots)\cdots,$$

and the least limit by

$$\underline{\lim}\, E_n = E_1E_2E_3\cdots + E_2E_3\cdots + E_3\cdots + \cdots.$$

If both formulas give the same result, $\lim E_n$ exists. (Cf. Vallée-Poussin, loc. cit. §10, pp. 8, 9.)

It follows that if the sets E_n are O of class α, $\overline{\lim}\, E_n$ is of the form a product of sums of sets which are O of class α. In the notation of §2 we have

$$\overline{\lim}\, E_n = \prod \sum O_\alpha = \prod O_\alpha = F_{\alpha+1}.$$

That is, $\overline{\lim}\, E_n$ is F of class $\alpha + 1$. Under the same hypothesis

$$\underline{\lim}\, E_n = \sum \prod O_\alpha = \sum F_{\alpha+1} = O_{\alpha+2}.$$

If the E_n are F of class α, similar reasoning shows that $\overline{\lim}\, E_n$ is F of class $\alpha + 2$ while $\underline{\lim}\, E_n$ is O of class $\alpha + 1$.

Consequently, if the E_n are ambiguous (A) of class α, $\overline{\lim}\, E_n$ is F of class $\alpha + 1$, $\underline{\lim}\, E_n$ is O of class $\alpha + 1$, and $\lim E_n$ when it exists is both O and F of class $\alpha + 1$, that is A of class $\alpha + 1$.

The following important theorem is readily inferred from properties of sets A of class α given by Vallée-Poussin (loc. cit., pp. 136–8); if $\alpha > 0$, every set A of class α is a limit of sets A of class $<\alpha$ and every set which is a limit of sets A of class $<\alpha$ is A of class α.

4. The characteristic function of a set of points. The function φ equal to one at the points of E and to zero elsewhere is called by Vallée-Poussin the *characteristic of E*. If the characteristic function φ is of class α, then E is A of class α. Conversely, if E is A of class α then φ is of class $\leqq \alpha$. If a set E is O or F of class α its characteristic is of class $\leqq \alpha + 1$.

If Q_1 and Q_2 have characteristics φ_1, φ_2, the characteristic of the product Q_1Q_2 is $\varphi_1\varphi_2$ while the characteristic of the sum $Q_1 + Q_2$ is given by the formula

$$\varphi_1 + \varphi_2 - \varphi_1\varphi_2.$$

This reduces to $\varphi_1 + \varphi_2$ when Q_1, Q_2 have no common element, that is, do not overlap. It follows that if φ_1, φ_2 are of class $\leqq \alpha$ then $Q_1 + Q_2$ is A of class α.

5. Relations between the classification of functions and the classification of sets of points. Let $(a < f < b)$, $(a \leqq f \leqq b)$ denote the sets of all points at which the respective inequalities hold for a given function f. Then if f is of class α, the set $(a < f < b)$ is O of class α, and the set $(a \leqq f \leqq b)$ is F of class α for every pair of values of a and b ($>a$). Furthermore there exist values of a and b such that the set $(a < f < b)$ is not O of any class $< \alpha$.

The functions of class α for which the set $(a < f < b)$ is always A of class α form an important subclass of the functions of class α denoted by the symbol A_α.

If a set Q is A of class α its complement $P - Q$ is also A of class α. It follows at once that a function of class α which assumes only a finite number of distinct values belongs to the class A_α, in fact to a subclass of A_α, which we denote by F_α.

If the values of a function f of class α form at most a denumerably infinite set $a_1, a_2, \ldots, a_n, \ldots$, which has no finite limit point, the function belongs to the class A_α. We denote by the symbol D_α the class of all such functions. The class D_α contains the class F_α.

II. On the composition and decomposition of sets of points in relation to the classification of Lebesgue

6. A theorem on the decompositon of sets of class $\alpha > 0$.

THEOREM. *If a set Q of class α is the sum of sets Q_1, Q_2 of class α which do not overlap there will exist sequences of sets K_n, K_{1n}, K_{2n} of class $< \alpha$ such that*

$$K_n = K_{\ n} + K_{2n}(n)$$

and $\lim K_n = Q$, $\lim K_{1n} = Q_1$, $\lim\ K_{2n} = Q_2$.

Since Q and Q_1 are of class α there exist sequences of sets H_n, H_{1n} of class $< \alpha$ which have the respective limits Q, Q_1. Set

$$K_n = H_n,\quad K_{1n} = H_n H_{1n},\quad K_{2n} = K_n - K_{1n}.$$

Then $K_n = K_{1n} + K_{2n}$ and the sets K_{1n}, K_{2n} are of class $< \alpha\ (n)$. If M is a point of Q_1 it is a point of Q and there will exist n_1 such that for all $n \geqq n_1$, M will belong to H_n and H_{1n}, therefore to K_{1n}. Consequently $\lim K_{1n} = Q_1$. If M is a point of Q_2 it is for all $n \geqq n_2$ an element of H_n and the complement of H_{1n}. Therefore M belongs to $H_n\ (P - H_{1n}) = K_n(P - H_{1n}) = K_n - K_{1n} = K_{2n}$. Hence, $\lim K_{2n} = Q_2$. This theorem may be generalized in an obvious way.

Let $\varphi_{1n}, \varphi_{2n}$ be the characteristics of K_{1n}, K_{2n}. The function $\varphi_n = a_1\varphi_{1n} + a_2\varphi_{2n}$ is of class $< \alpha$ and converges with increasing n to the function $\varphi = a_1\varphi_1 + a_2\varphi_2$ of class α where φ_1 and φ_2 are the respective characteristics of Q_1, Q_2.

7. Theorems on decomposition.

THEOREM I. *Let $E_1, E_2, \ldots, E_n$, beany system of sets O of class $\alpha > 0$ such that*

$$P = E_1 + E_2 + \cdots + E_n.$$

Then there exists a normal system $H_1, H_2, \ldots, H_n$ of sets A of class α such that:

(1) *the sets, $H_1, H_2, \ldots, H_n$ do not overlap;*

(2) *the set* H_i *is contained in* E_i $(i = 1, 2, \ldots, n)$;

(3) $P = \sum_{i=1}^{n} H_i$.

Since each set E_i is O of class α it may be represented in the form $E_i = \sum_{m-1}^{\infty} F_{im}$ where F_{im} is A of class α (m) (cf. Vallée-Poussin, loc. cit., p. 138, 5°), and we may assume without loss of generality that $F_{i,m+1}$ contains F_{im}. Set $H_{11} = F_{11}$, $H_{i0} = [0]$, and

$$H_{im} = F_{im}P\Big(-\sum_{j=1}^{j=i-1} H_{jm} - \sum_{j=i+1}^{j=n} H_{j,m-1} \Big).$$

Then H_{im} is A of class $\alpha(im)$. We wish to establish the following properties of the system $((H_{im}))$:

(A) H_{im+1} contains H_{im} (im);

(B) if i and j are unequal, H_{im} and H_{jn} have no element in common for any pair of values of m and n;

(C) every element of P is contained in some class H_{im}.

It is clear that $H_{i1} = F_{i1}(P - \sum_{j=1}^{i-1} F_{j1})$, and that in consequence the H_{j1} do not overlap. Since $H_{12} = F_{12}(P - \sum_{j=2}^{n} H_{j1})$ it is evident that H_{12} contains H_{11}, and it is likewise easy to see that H_{i2} contains H_{i1} and that the H_{i2} do not overlap. It is now simply a matter of induction to establish property (A) and to show that for fixed m the H_{im} do not overlap. To establish the remaining part of (B), we observe that if m is greater than n, then H_{jm} contains H_{jn} and has no element in common with H_{im}. Finally, by hypothesis, every element of P lies in some E_i and therefore in some F_{im}. If this element does not belong to H_{im}, it is because it belongs to one of the classes $H_{jm}(j < i)$ or $H_{j,m-1}(j > i)$.

We now set $H_i = \sum_{m=1}^{\infty} H_{im}$. It follows from the properties of the system $((H_{im}))$ just established that the system (H_i) satisfies the conditions (1, 2, 3) of the theorem. But H_i is certainly O of class α and its complement is therefore F of class α. But the complement of H_i is $\sum_{j=1}^{i-1} H_j + \sum_{j=i+1}^{n} H_j$ and is consequently O of class α. It follows that H_i is A of class $\alpha(i)$. This completes the proof of the theorem.

Theorem II. *If $P = \sum_{n=1}^{\infty} E_n$, E_n is O of class α for every n, and $\prod_{n=1}^{\infty} E_n = [0]$, there exists a sequence of classes F_n, A of class α (n), such that $F_n F_{n'} = [0]$ $(n' \neq n)$, $\sum_{n=1}^{\infty} F_n = P$, and $F_n \leqq E_n (n)$.*

From the preceding theorem we have, if we denote $\sum_{m=n+1}^{\infty} E_m$ by R_n, a decomposition of

$$P = E_1 + E_2 + \cdots + E_n + R_n$$

into a normal system:

$$P = F_{n1} + F_{n2} + \cdots + F_{nn} + G_n.$$

Then if we set

$$F_1 = F_{11},$$
$$F_2 = (F_{12} + F_{22})\,(P - F_1),$$

and in general,

$$F_n = \left(\sum_{i=1}^{n} F_{in}\right)\left(P - \sum_{i=1}^{n-1} F_i\right),$$

we have the desired sequence $\{F_n\}$.

III. Functions of class α — reduction — superposition

8. A generalization of a theorem of Vallée-Poussin. The following theorem is a direct generalization of a theorem stated by Vallée-Poussin for functions of class 1.

Theorem. *Every function of class $\alpha > 0$ is the limit of a uniformly convergent sequence of functions of class D_α.*

In fact if we let $E_n = (\overline{n-1}e < f < \overline{n+1}e)$, where e denotes an arbitrarily small positive number, the set E_n is O of class α and $P = \sum_{n=-\infty}^{+\infty} E_n$. Applying the results of the preceding article we obtain a sequence of sets F_n, A of class α, which do not overlap, such that F_n is contained in E_n and $\sum_{n=-\infty}^{+\infty} F_n = P$. If we set $\varphi = ne$ on F_n (n), φ is defined on P, is of class D_α, and it is evident from the inclusion of F_n in E_n that $|\varphi - f| \leqq e$.

The theorem follows immediately.

9. A theorem on superposition of functions of class A_α.

Theorem. *If $Q_1, Q_2, \ldots, Q_m, \ldots$ is a sequence of non-overlapping sets whose characteristics φ_m are of class $\leqq \alpha$, $f_1, f_2, \ldots, f_m, \ldots$, is a sequence of*

functions of class A_α *at most, and* $P = \sum_{m=1}^{\infty} Q_m$, *then*

$$f = \sum_{m=1}^{\infty} \varphi_m f_m$$

is of class A_α *on* P.

Consider the set $E = (a \leqq f \leqq b)$. It is of the form $E = \sum_{m=1}^{\infty} U_m$, where $U_m = (a \leqq \varphi_m f_m \leqq b)$ and is A of class α (m). But $Q_m - U_m$ is A of class $\alpha(m)$ and therefore both $P - E = \sum_{m=1}^{\infty}(Q_m - U_m)$ and E are O of class α. It follows that E is A of class α, which was to be proved.

10. Reduction-functions of class α on a subset of P. If a function f is defined on P its reduction $f(Q)$ relative to a subset Q of P is the function on Q which has the functional values of f.*

We shall find it convenient to classify functions on an arbitrary subset Q of the fundamental set.

A function f is of class α on Q in case it is not of class $< \alpha$ on Q and is the reduction relative to Q of a function ψ which is the limit on P of a sequence of functions of class $\leqq \alpha$ on P converging uniformly on Q.

It is evident that the reduction relative to Q of a function of class α on P is of class $\leqq \alpha$ on Q, but that a function of class α on Q may not be the reduction of a function of class α, although it must be the reduction of a function of class $\alpha + 1$. It will be noticed that the reduction of a function of class α may be of lower class; for example, the reduction relative to Q of the characteristic of Q is of class zero. Not every continuous function on an open set is of class zero.† For example, let Q denote the sequence $1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots$, and let $f(\frac{1}{n})$ equal 1 (n odd), 2 (n even). Then $f(\frac{1}{n})$ is of class 1, although continuous on Q.

It follows readily from a theorem of Vallée-Poussin (loc. cit., § 130, p. 130) that the class α of functions on Q as defined above has the usual closure relative to uniform convergence. The functions of class $\alpha + 1$ whose reductions relative to Q are of class α on Q form a subclass of the class $\alpha + 1$.

The definition of function of class α on Q given above is to be compared with the following definition of Vallée-Poussin (loc. cit., § 135, p. 139): *A function f is within e of class $\alpha > 0$ on a set Q if, on Q, f is within e of being a limit of functions of class $< \alpha$ on P.*

The class of functions, within e of class α on Q for every $e > 0$, contains the functions of class α on Q as defined above. If Q is a closed set these two classes

* E. H. Moore, loc. cit., p. 91, § 52.

† It is easy to see that every function which is continuous on an arbitrary set Q is the reduction of a semi-continuous function, that is, a function of class A_1.

of functions are the same for every value of $\alpha > 0$. It would be of interest to know whether there exist classes Q for which this conclusion fails to hold.

11. THEOREM. *Let Q be an arbitrary subset of P and suppose that $f_1, f_2, \ldots, f_n \ldots$, is a sequence of functions of class $\leqq \alpha$ converging uniformly on Q. Then Q is contained in a set $\overline{Q}$ which is F of class α and the sequence converges uniformly on $\overline{Q}$.*

Because of the uniform convergence on Q we have for every integer m and integer n_m such that for all $n \geqq n_m$ the inequality

$$|f_n - f_{n+p}| \leqq \frac{1}{m}$$

holds on Q for every positive integer p. Let Q_{np}^m be the set of all points of P at which the above inequality holds when m, n, p are fixed. Then Q_{np}^m is F of class α and contains Q. The product $\prod_{p=1}^{\infty} Q_{np}^m$ is likewise F of class α and must contain Q. Denote by Q_m the product of the sets

$$Q_{n_m}^m Q_{n^m+1}^m \cdots Q_{nm+1}^m.$$

The product $\overline{Q} = \prod_{m=1}^{\infty} Q_m$ is the set required by the theorem. We must show that the convergence is uniform on $\overline{Q}$. It is of course obvious that $\overline{Q}$ is F of class α and contains Q.

Let e be an arbitrary small positive number and choose m so that $\frac{1}{m} \leqq e$. Then for all points of Q_m, we have, whenever $n \geqq n_m$,

$$|f_n - f_{n+p}| \leqq e \ (p = 1, 2, 3, \ldots).$$

But $\overline{Q}$ is contained in $Q_m(m)$. Therefore this inequality holds on $\overline{Q}$ for every e. This establishes the desired uniformity.

COROLLARY I. *If a sequence of continuous functions converges uniformly on a set Q it converges uniformly on $Q°$, the set obtained by adding its limiting points to the set Q.*

This is the case $\alpha = 0$, and the sets which are F of class zero are closed. The set $Q°$ is contained in every closed set which contains Q. The corollary may be proved directly from the fact that if a function f is continuous and $|f| \leqq e$ at the points of Q, then the same inequality is true for all points of $Q°$.

COROLLARY II. *In case $\alpha = 0$, the limit function is continuous on Q and $Q°$ and the function on Q is the reduction relative to Q of a continuous function.*

12. THEOREM. *If a sequence $f_1, f_2, \ldots, f_n, \ldots$ of functions of class $\leqq \alpha$ converges uniformly on a set Q which is F of class α to a limit function f, the set QE, where $E = (a \leqq f)$ is F of class α.*

Because of the uniform convergence on Q, we have for every positive integer

m a positive integer m_n such that for all points of QE, $f_{nm} \geqq a - \frac{1}{m}$. Let $E_m = (f_{nm} \geqq a - \frac{1}{m})$ and set $Q_m = QE_m$. Then Q_m is F of class α and contains QE. In fact $QE = \prod_{n=1}^{m} Q_n$, since if M is a point common to all Q_m then

$$f(M) = \lim_{M \to \infty} f_{nm}(M) \geqq a$$

and M is by definition a point of E.

13. Theorems on superposition generalizing theorems of Baire.*

THEOREM I. *If f_1 and f_2 are of class $\leqq \alpha$ and Q_1, Q_2 are A of class α the function f equal to f_1 on Q_1 and to f_2 on $Q_2 - Q_1Q_2$ is of class $\leqq \alpha$ on $Q_1 + Q_2$.*

The characteristics φ_1, φ_2 of Q_1, $Q_2 - Q_1Q_2$ are of class $\leqq \alpha$. The function

$$f = \varphi_1 f_1 + \varphi_2 f_2$$

is of class $\leqq \alpha$ and satisfies the conditions of the theorem. The theorem may obviously be extended to any finite number of sets and functions.

THEOREM II. *If $f_0, f_1, f_2, \ldots, f_n, \ldots$, is a sequence of functions of class $\leqq \alpha$ on $P = P_0$ and if $P_1, P_2, \ldots, P_n, \ldots$ are subsets of P and A of class $< \alpha$, the function f equal to f_n on the set $P_n(P_0 - \sum_{i=1}^{n} P_i)$ and to f_0 on $P_0 - \sum_{n=1}^{\infty} P_i$ is of class $\leqq \alpha$.*

Evidently the characteristic φ_n of $P_n(P_0 - \sum_{i=1}^{n-1} P_i)$ is of class $< \alpha$. For each value of n there is a sequence of functions f_{km} such that $\lim_{k \to \infty} f_{kn} = f_n$. The function φ_ν equal to $f_{\nu n}$ on $P_n(P_0 - \sum_{i=1}^{n-1} P_i)$ for $n = 1, 2, \ldots, \nu$, and to $f_{\nu 0}$ on $P_0 - \sum_{i=1}^{\nu} P_i$ is, by an obvious extension of the preceding theorem, of class $< \alpha$. But $\lim_{\nu \to \infty} \varphi_\nu = f$, and therefore f is of class $\leqq \alpha$.

IV. CONVERGENCE (R) — LIMIT FUNCTIONS OF SEQUENCES CONVERGENT (R) — CLASSIFICATION OF FUNCTIONS IN TERMS OF CONVERGENCE (*R*)

14. Definition of convergence (*R*). Theorem of equivalence.

According to the definition of E. H. Moore a sequence of functions f_n ($n = 1, 2, 3, \ldots$) is relatively uniformly convergent, that is, convergent (R), on a range P (which may be an arbitrary set) if there exist functions f, σ defined on

* Acta Mathematica, vol. 30 (1906), pp. 16, 17, § 15; p. 31, § 28.

P, and for every small positive number e an integer n_e, such that the inequality

$$|f_n - f| \leqq e|\sigma|$$

holds for all elements of P. If the scale function σ is identically 1 the convergence becomes uniform convergence. If σ is bounded from zero and infinity the convergence is equivalent to uniform convergence. If σ is a limited function and not bounded from zero the convergence is subject to a stronger condition than that of uniform convergence. Finally if σ is unlimited and $|\sigma| > a > 0$ the condition is weaker than the condition of uniform convergence. It is possible to define convergent sequences which do not converge uniformly relative to any scale function σ.*

The following relation between uniform and relatively uniform convergence is of fundamental importance.

THEOREM. *A necessary and sufficient condition that a convergent sequence of functions f_n converge relatively uniformly on P to a function f is that there exist a sequence of classes Q_m such that Q_{m+1} contains Q_m, every element of P lies in some class Q_m and the sequence converges uniformly on Q_m (m).*†

The condition is necessary. From the existence of the scale function σ we obtain Q_m as the set ($|\sigma| \leqq m$).

The condition is sufficient. Let φ denote the function whose value is equal to the greatest of the values of the functions in the convergent sequence, $|f_1|, |f_2|, \ldots, |f_n| \ldots$. The function φ is everywhere defined and finite, and is in fact the smallest function for which all of the inequalities

$$|f_n| \leqq \varphi \; (n=1, 2, 3, \ldots)$$

hold uniformly on P.

15. Theorems on convergence (R).

THEOREM I. *If* $f = \lim\limits_{n\to\infty} f_{1n}$ *and* $f_2 = \lim\limits_{n\to\infty} f_{2n}$ *and the convergence is relatively uniform in each case then* $f_1 + f_2 = \lim\limits_{n\to\infty} f_{1n} + f_{2n}$ *and the convergence is relatively uniform.*

In fact if the scale functions are, respectively, σ_1, σ_2, then $\sigma = |\sigma_1| + |\sigma_2|$ is effective as a scale function in the case of the sequence of functions, $f_{1n} + f_{2n}$.

It is evident that this theorem may be extended to the case of any finite number of functions.

* These Transactions, vol. 15, p. 201.

† "Relatively uniform convergence" and "relatively uniform convergence with respect to σ" are different concepts. In one case there exists a scale function σ, in other the scale function is prescribed.

E. H. Moore has called attention to the following. *If* $f_m = \lim_{n\to\infty} f_{mn}$ *and the convergence is uniform relative to* $\sigma_m (m = 1, 2, \ldots)$, *and if there is a function* σ *and a sequence of positive numbers* a_m *such that* $|\sigma_m| \leqq a_m|\sigma|$ *identically for every* m, *then the convergence to* f_m *is in every case uniform relative to* σ.*

The function σ is said to be a dominant of the sequence of functions σ_m.

THEOREM II. *If* $f = \lim_{m\to\infty} f_m$ *uniformly relative to* σ_0, *and* $f_m = \lim_{n\to\infty} f_{mn}$ *uniformly relative to* $\sigma_m (m = 1, 2, \ldots,)$ *and if* σ *is a dominant of the sequence* $\sigma_0, \sigma_1, \ldots, \sigma_m, \ldots$, *then a subsequence of the sequence of functions* f_{mn} *converges to* f *uniformly relative to* σ.

From the theorem of Moore just stated each convergence is uniform relative to σ. Choose f_{mn_m} so that $|f_m - f_{mn_m}| \leqq \frac{1}{m}|\sigma|$. Then the functions f_{mn_m} converge to f uniformly relative to σ.

THEOREM III. *If* $E_1, E_2, E_3, \ldots$ *are non overlapping sets, and* $f_1, f_2, f_3, \ldots$ *is any sequence of functions, the series*

$$\sum_{m=1}^{\infty} f_m \varphi_m,$$

where φ_m *is the characteristic of* E_m, *converges relatively uniformly on* P.

Let $E = \sum_{m=1}^{\infty} E_m$. Let $Q_m = \sum_{i=1}^{m} E_i + P - E$. Then $\lim_{n\to\infty} Q_m = P$ and the convergence is uniform on Q_m (m).

THEOREM IV. *If a sequence* $f_1, f_2, f_3, \ldots$ *of functions converges* (R) *to a function* f *and if* φ *is any uniformly continuous function over the interval of variation of the functions* $f, f_1, f_2, \ldots$, *then* $\varphi(f) = \lim_{n\to\infty} \varphi(f_n)$ *and the convergence is relatively uniform.*

For f_n converges to f uniformly on each of a sequence of sets $Q_1, Q_2, Q_3, \ldots, Q_m, \ldots$ such that Q_m contains Q_{m-1} and $\lim_{n\to\infty} Q_m = P$. It follows at once that $\varphi(f_n)$ converges uniformly to $\varphi(f)$ on Q_m. Therefore the convergence is relatively uniform.

16. On the limits of relatively uniformly convergent sequences of functions of class $\leqq \alpha$.

THEOREM I. *If* f *is the limit of a sequence of functions* f_n *of class* $\leqq \alpha$ *converging* (R) *there exists a sequence* $Q_1, Q_2, Q_3, \ldots,$ *of subsets of* P *such that*: Q_{m+1} *contains* Q_m, Q_m *is* F *of class* α; *the convergence is uniform on* Q_m; f *is of class* $\leqq \alpha$ *on* $Q_m(m)$; *and* $\lim_{m\to\infty} Q_m = P$.

This theorem is an immediate consequence of the theorems of §12 and §14.

* E. H. Moore, loc. cit., p. 50, § 25, Theorem I.

COROLLARY I. *Every sequence of functions of class* $\leqq \alpha$, *which converges* (R), *converges uniformly relative to a scale function of the class* $A_{\alpha+1}$ *at most.*

COROLLARY II. *If the sets* Q_m *are* A *of class* α *there is an effective scale function of class* D_α *at most.*

THEOREM II. *A necessary and sufficient condition that a function* f *be the limit of a sequence of functions of class* $\leqq \alpha$ *convergent* (R) *is that there exist a sequence*

$$Q_1, Q_2, Q_3, \ldots,$$

of subclasses of P *such that* $\sum_{m=1}^{\infty} Q_m = P$, *and the reduced function* $f(Q_m)$ *shall be of class* $\leqq \alpha$ (m).

The necessity of the condition follows from the preceding theorem. To prove the condition sufficient we note that there is for every positive integer m a function f_m of class $\leqq \alpha$ such that on $\sum_{i=1}^{m} Q_i$, $|f - f_m| \leqq \frac{1}{m}$. The sequence of functions f_m converges uniformly on $\sum_{i=1}^{n} Q_i$ for every value of n and is therefore convergent (R) by the theorem of § 14.

THEOREM III. *If a sequence of functions* f_n *of class* $\leqq \alpha$ *is convergent* (R) *to* f *every set* $E = (a \leqq f)$ *is the limit of sets* F *of class* α. *Furthermore, there exists (for every value of* a*) a sequence of sets* Q_m *which are* F *of class* α *and have the sum* P, *such that* EQ_m *is* F *of class* α *for every value of* m $(m = 1, 2, 3, \ldots)$.

The corresponding statements may be made for the sets $E = (f \leqq a)$.

From the theorem of § 16 there exists a sequence of sets Q_m with limit P such that each set Q_m is F of class α and the convergence is uniform on Q_m. From the theorem of § 13 the sets Q_mE are F of class α. Since $\lim_{m\to\infty} Q_mE = E$, the theorem is proved.

COROLLARY. *The limit of a sequence of functions of class* $\leqq \alpha$ *convergent* (R) *is of class* $A_{\alpha+1}$ *at most.*

For the sets EQ_m are F of class α and therefore O of class $\alpha+1$. But $E = \sum_{m=1}^{\infty} EQ_m$ and is therefore O of class $\alpha + 1$. Since the limit function is of class $\alpha+1$ at most, the sets E are F of class $\alpha+1$, therefore A of class $\alpha+1$, which implies the conclusion stated.

17. Theorems on the limit of a sequence of functions of class $<\alpha$ **convergent** (R).

In case α is of the second kind we obtain a more precise result than is obtained from the statement of the theorem of § 16.

THEOREM I. *If a function f is the limit of a sequence of functions f_n of class $< \alpha$ convergent* (R), *every set* $E = (a \leqq f)$ *is A of class α.*

Since the convergence is uniform on each of a sequence of sets Q_m we may find n_{km} such that on Q_m,

$$|f_n - f_{n+h}| \leqq \tfrac{1}{k} \quad (k=1, 2, 3, \ldots),$$

whenever n exceeds n_{km}.

Let Q_{nh}^{km} denote the set of all points of P at which the preceding inequality holds. Evidently Q_{nh}^{km} is A of some class $\beta < \alpha$. From the quadruply infinite system $((Q_{nh}^{km}))$ we may select a subsystem (T^{km}) such that the product T^m of the T^{km} contains Q_m while the sequence f_n converges uniformly on T^m. This is an obvious consequence of the fact that the inequality above holds on T^{km} which must contain Q_m. If we set $E_{nk} = (a - \frac{1}{k} \leqq f_n)$ then E_{nk} is A of some class $\beta < \alpha$. Therefore $T^{km}E_{nk}$ is A of some class $\beta < \alpha$. But $T^mE = \prod_{k=1}^{\infty} T^{km}E_{n_k k}$ where n_k is chosen so that E_{n_k} contains Q_mE, which choice can be made because of the uniform convergence on Q_m. It follows that T^mE is A of class α and so O of class α. But $E = \sum_{m=1}^{\infty} T^mE$ is of the form $\sum O_\alpha$, that is, E is O of class α. But E is also F of class α, since f is of class α and $E = (a \leqq f)$. Therefore E is A of class α.

From the preceding theorems and the corollary to a theorem of § 16 we obtain the important result:

THEOREM II. *If a sequence of functions f_n of class $< \alpha$ converges to a limit function f of class α relatively uniformly, then f belongs to the class A_α.*

18. Sequences of characteristic functions, convergent (R).

THEOREM I. *If* $Q = \lim_{m \to \infty} Q_m$ *then the characteristics φ_m of the sets Q_m converge relatively uniformly to the characteristic of Q.*

In fact on $QQ_m + (P - Q)(P - Q_m)$ we have $\varphi = \varphi_m$, where φ denotes the characteristic of Q. Let $V_m = \prod_{n=m}^{\infty} [QQ_m + (P - Q)(P - Q_m)]$. It is at once evident that the convergence is uniform on $V_m(m)$, and that V_m contains V_{m-1}. We will show that every element of P lies in V_m for a sufficiently large value of m. In fact if M is a point of Q there is, since $Q = \lim_{m \to \infty} Q_m$, an integer m_0 such that for all $m \geqq m_0$, M belongs to Q_m and therefore to QQ_m, and so to V_m. If M is a point of $P - Q$ a similar argument prevails.

THEOREM II. *Every function of class D_α is a limit of a relatively uniformly convergent sequence of functions of class $< \alpha$.*

Suppose f of class D_α is equal to a_m on E_m, ambiguous of class α. Let φ_m be the characteristic of E_m and consider the function

$$g_m = \sum_{k=1}^{m} a_k \varphi_k .$$

Since E_m is the limit of a sequence of sets A of class $< \alpha$ it follows that φ_k $(k \leqq m)$ is the limit of a sequence φ_{kn} of characteristic functions of class $< \alpha$ convergent (R). Therefore g_m is the limit as n becomes infinite of the sequence $g_{mn} = \sum_{k=1}^{m} a_k \varphi_{kn}$, convergent (R). We proceed to show that a sequence taken from the doubly infinite set of functions g_{mn} converges (R) to f.

Since $E_m = \lim_{n \to \infty} E_{mn}$, where E_{mn} is A of class $< \alpha$, the functions φ_{mn} are equal to 1 and to φ_m on $H_{mk} = \prod_{i=k}^{\infty} E_{mi}$ whenever n exceeds k. Evidently $E_m = \lim_{k \to \infty} H_{mk}$. It follows that

$$f = g_m = g_{mn}$$

on $Q_m = \sum_{k=1}^{m} H_{km}$. But $\lim_{m \to \infty} Q_m = P$. For if M is a point of P it lies in some class E_{m_0}, therefore in some set $H_{m_0 k_0}$. Then if $m_0 \geqq k_0$, the point M belongs to Q_{m_0}. If $m_0 < k_0$, the point M belong to Q_{k_0}. Therefore the sequence g_{mm} converges (R) to the function f.

19. The classification of function in terms of convergence (R).

We have seen that the limit functions of sequences of functions of class $\leqq \alpha$ which are convergent (R) belong to class $A_{\alpha+1}$ at most. Denoting the class of all functions of class α by B_α we see that while B_α is not closed for relatively uniform convergence its extension in terms of relatively uniform convergence is a subclass of $B_\alpha + A_{\alpha+1}$.

If we set $R_0 = B_0$; R_1 equal to the class of all functions which are not in R_0 and are the limits of relatively uniformly convergent sequences of functions of R_0; and define R_α in similar fashion in terms of the ordinals $< \alpha$; then it is evident that

$$D_{\alpha+1} < R_{\alpha+1} \leqq B_\alpha + A_{\alpha+1}.$$

In case α is of the second kind we have

$$D_\alpha < R_\alpha \leqq A_\alpha < B_\alpha .$$

We have established previously* that

$$R_1 = A_1 .$$

* These Transactions, vol. 20 (1919), p. 184.

University of Iowa,
Iowa City, Iowa